Exercise 82

A tangent line is drawn to the hyperbola $xy = c$ at a point P.

(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P.

(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

Solution

Solve the equation of the hyperbola for y.

$$y = \frac{c}{x}$$

Take the derivative.

$$y' = \frac{d}{dx}\left(\frac{c}{x}\right) = c\frac{d}{dx}(x^{-1}) = c(-x^{-2}) = -cx^{-2}$$

Let the coordinates of point P be (x_p, y_p). That means the hyperbola's slope at this point is

$$y'(x_p) = -cx_p^{-2}.$$

The equation of the tangent line with this slope going through the point P is

$$y - y_p = -cx_p^{-2}(x - x_p).$$

Set $y = 0$ to determine the x-intercept of the line.

$$-y_p = -cx_p^{-2}(x - x_p) \quad \rightarrow \quad x = \frac{x_p(c + x_p y_p)}{c} \quad \Rightarrow \quad \left(\frac{x_p(c + x_p y_p)}{c}, 0\right)$$

Set $x = 0$ to determine the y-intercept of the line.

$$y - y_p = -cx_p^{-2}(-x_p) \quad \rightarrow \quad y = \frac{c + x_p y_p}{x_p} \quad \Rightarrow \quad \left(0, \frac{c + x_p y_p}{x_p}\right)$$

Use the midpoint formula to determine the point halfway between these intercepts.

$$M = \left(\frac{\frac{x_p(c+x_p y_p)}{c} + 0}{2}, \frac{0 + \frac{c+x_p y_p}{x_p}}{2}\right)$$

For any point on the hyperbola, $xy = c$, which means $x_p y_p = c$.

$$M = \left(\frac{\frac{x_p(c+c)}{c} + 0}{2}, \frac{0 + \frac{x_p y_p + x_p y_p}{x_p}}{2}\right) = (x_p, y_p)$$

Therefore, the midpoint of the line segment cut from this tangent line by the coordinate axes is P. The area of the triangle formed by the tangent line and the coordinate axes is

$$A = \frac{1}{2}bh = \frac{1}{2}\left[\frac{x_p(c + x_p y_p)}{c}\right]\left[\frac{c + x_p y_p}{x_p}\right] = \frac{1}{2c}(c + x_p y_p)^2 = \frac{1}{2c}(c + c)^2 = 2c.$$

Therefore, the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.